GEOMETRY OF DIFFERENTIABLE MANIFOLDS WITH INDEFINITE METRICS

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PUJA DHASMANA

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
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PREFACE

During the last decade study of Differentiable Manifolds with indefinite metric has attracted the Mathematicians partly because of its elegance and partly because of its applications in General Relativity and Relativistic Physics.

Pseudo Riemannian Manifolds have been the subject of study since the time of Elie Cartan. It was only after the work of Bognar (1976) and others on indefinite inner product spaces, and the pressing need of putting the general Relativity theory on a more sound mathematical ground that the work on pseudo Riemannian Manifolds and in particular on Lorentz Manifold got an empetus.

The present dissertation comprises four chapters of which Chapter I is introductory and gives brief resume of the results on the theory of Pseudo Riemannian Manifolds and submanifolds with indefinite metric which are of relevance in our subsequent chapters.

Chapter II deals with Einstein Hypersurfaces of indefinite space form. A submanifold M of differentiable manifold $\bar{M}$, is called hypersurface if it is of codimension one, and is called Einstein if Ricci tensor S is multiple of metric tensor g. Mostly the results in this chapter are due to Martin,

A. Magid (cf. [16], [17], [18]). The Geometry of hypersurfaces depends on the shape operator and therefore the different possibilities of the shape operator are discussed. Specially we have discussed indefinite hypersurface with shape operator $A^2 = 0$ and $A^2 = -b^2 Id$, where b is non zero real constant.

In Chapter III, Indefinite Hypersurface of Lorentz space form have been studied. The main results in this chapter are due to Martin A. Magid [19], in which he has studied special classes of parallel submanifolds in R_1^m, Lorentz space of signature (1, m-1) and in R_2^m, Eucledian space of signature (2, m-2). Also we have classified umblical submanifold as well as isometric immersion R_1^n —— R_1^{n+k}, R_1^n —— R_1^{n+2} with parallel second fundamental form.

In Chapter IV, Nullity distribution of indefinite Immersion have been studied. The main results in this chapter are due to Kinetsu Abe and Martin Magid [1], where they have investigated the relative nullity distribution of an indefinite Riemannian manifold isometrically immersed into an indefinite space form. It has been noted that this distribution is totally geodesic and give rise to a Ricatti type differential equation along the geodesic in a leaf.

CHAPTER-I

INTRODUCTION

In this chapter, to make the dissertation self contained we describe some results in the geometry of Riemannian manifolds which are relevent to the discussion in the forth coming chapters.

1.1. Structures on Manifolds

Let M be a smooth manifold and $p \in M$. A scalar product g_p(non degenerate, symmetric bilinear form) of index ν on tangent space $T_p(M)$ which also depends differentiably on M, i.e.,

$$g_p : T_p(M) \times T_p(M) \xrightarrow[\text{Symm, nondeg.}]{\text{bilinear}} R$$

is a tensor of type (0,2). The corresponding tensor field g is called the metric tensor.

Definition 1.1.1 : Let (M,g) be Riemannian manifold. The index ν of g on M is the largest integer, i.e., the dimension of subspace $N \subset M$ on which $g|N$ is Negative definite.

Definition 1.1.2 : Smooth manifold M equipped with metric tensor g defined above is called Pseudo Riemannian manifold (Indefinite Riemannian manifold).

Now, if $\nu = 1$, and $n \geq 2$, then (M,g) is called Lorentz Manifold.

Definition 1.1.3 : Let X be a tangent vector on M. Then

i) X is space like if $g(X,X) > 0$,

ii) X is time like if $g(X,X) < 0$,

iii) X is null vector if $g(X,X) = 0$, $X \neq 0$.

The set of all null vectors in $T_p(M)$, $p \in M$, is called a null cone at p.

Definition 1.1.4 : The metric g is said to be nondegenerate if

$$g(X,Y) = 0 \implies X = 0,$$

for all non-zero Y.

Definition 1.1.5 : A subspace S of $T_p(M)$ is said to be nondegenerate (degenerate) if the restriction of g to S is nondegenerate (degenerate). Now we have the following characterization for the plane to be non-degenerate.

Lemma 1.1.1 : If $\{X,Y\}$ is a basis of plane P i.e. (two dimensional subspace of tangent space) then P is nondegenerate iff

$$g(X,X)\ g(Y,Y) - g(X,Y)^2 \neq 0 .$$

<u>Proof</u>: Let ξ be a fixed vector,

$$\xi = \alpha X + \beta Y ,$$

consider an arbitrary vector $Z = xX + yY$ in the plane P. Then P is nondegenrate if

$$g(\quad ,Z) = 0 \quad \forall\, Z \Longrightarrow \quad = 0 ,$$

$$0 = g(\quad ,Z) = g(\alpha X+\beta Y,\ xX+yY)$$

$$= \{\alpha g(X\ X)+ \beta g(Y,X)\}\, x + \{\alpha g(X,Y)+\beta g(Y,Y)\}\, y ,$$

where x and y are arbitrary .

This implies that

$$\alpha g(X,X) + \beta g(Y,X) = 0 ,$$

$$\alpha g(X,Y) + \beta g(Y,Y) = 0 .$$

These equations would have solution $\alpha = o,\ \beta = o$ iff

$$g(X\ X)\ g(Y,Y) - g(X,Y)^2 \neq 0 .$$

Q.E.D.

<u>Definition</u> 1.1.6 : For non-degenerate plane P, the sectional curvature $K(P)$ is defined by

$$K(P) = \frac{\langle R(X,Y)\ Y,X\rangle}{g(X\ X)\ g(Y,Y) - g(X,Y)^2} ,$$

Where $\{X,Y\}$ is a basis of plane P. The definition becomes meaningless if the plane P is degenerate.

<u>Definition</u> 1.1.7 : Let (M,g) be an n-dimensional Pseudo Riemannian manifold with connection D, then the Ricci tensor S of M is defined by

$$S(X,Y) = \sum_{i=1}^{n} \sigma_i \, g(R(e_i,X)Y,e_i),$$

where $e_1,\ldots,e_n$ is an orthonormal frame of $T_x(M)$ and $g(e_i,e_j) = \sigma_i \delta_{ij}$, $\sigma_i = \pm 1$.

<u>Definition</u> 1.1.8 : Let M be an indefinite Riemannian manifold. M is said to be Einstein if the Ricci tensor S is multiple of metric tensor g defined before.

<u>Definition</u> 1.1.9 : Let M be differentiable manifold of even dimension. An almost complex structure on M is a tensor field J of type $(1,1)$ such that $J^2X = -X$ for each vector field X on M, i.e., $J^2 = -Id$.

An almost complex manifold is a pair (M,J) where M is a C^∞-manifold and J is an almost complex structure on M.

On almost complex manifold there always exist a Riemannian metric g consistent with almost complex structure J, i.e. it satisfies the condition.

$$g(JX, JY) = g(X,Y), \quad X,Y \in \mathcal{X}(M),$$

by virtue of which g is called a Hermitian metric.

Definition 1.1.10 : Let ∇ be the Riemannian connection on an almost complex manifold (M,J) with metric g such that

$$\nabla J = 0 \text{ , i.e., } J \text{ is parallel}$$

Then M is called a Kaehler manifold.

1. 2. Space forms :

Let (M,g) be Riemannian manifold. The curvature tensor R of M is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z ,$$

Where ∇ is Riemannian connection, $[\ ,\ ]$ denote the lie bracket.

Definition 1.2.1 : The sectional curvature of the plane (by plane we mean 2-dimensional subspace of tangent space) determined by orthonormal vectors X,Y, is defined

$$K(X,Y) = g\ (R(X,Y)\ Y,X) .$$

If at each point the sectional curvature function turns out to be constant c, then M is said to be Real Space Form and in which case Riemannian curvature function admits the simple form

$$R(X,Y)Z = c\ [g(Y,Z)X - g(X,Z)Y] .$$

In case of Kaehler manifold, $\{X,JX\}$ turns out to be an orthonormal frame for every unit vector X, and therefore, generate a plane. The sectional curvature $K(X,JX)$ is denoted by $H(X)$ and is called <u>Holomorphic sectional curvature</u>.

If Holomorphic sectional curvature $H(X)$ is constant at every point $m \in M$, then M is said to be space of constant Holomorphic sectional curvature.

<u>Definition</u> 1.2.2 : A Kaehler manifold of constant holomorphic sectional curvature is called <u>Complex Space form</u>.

<u>Definition</u> 1.2.3 : In case of complex space form Riemannian curvature tensor is given by

$$R(X,Y)Z = \frac{c}{4}\,[g(Y,Z)X - g(X,Z)Y + g\,(JY,Z)JX$$

$$- g(JX,Z)JY + 2g(X,JY)\,JZ]$$

for X,Y,Z tangent to M.

1.3. <u>Submanifolds</u> :

To study the geometry of manifold, sometimes it becomes more convenient to first embed it into a manifold of which geometry is known and then watch the geometry which is induced on it. In fact the classical study of geometry started with theory of curves and surfaces in R^3 (i.e. embeded in R^3). This strengthens the view that the study of submanifolds is a basic branch of geometry.

Let f be a differentiable map form manifold M into manifold $\bar{M}$, and let the dimension of M and $\bar{M}$ be n and m. If at each point p of M, $(f_*)_p$ is one-one (i.e. if $\text{rank}_p f = n$), then f is called an immersion of M into $\bar{M}$.

If f is an immersion and if moreover f is one-one map i.e. $f(p) \neq f(q)$ for $p \neq q$ then f is called an embedding of M into $\bar{M}$.

If an n-dimensional differentiable manifold M admits an immersion $f : M \longrightarrow \bar{M}$ into an m-dimensional differentiable manifold $\bar{M}$. Then M is said to be a submanifold of $\bar{M}$. Naturally $n \leq m$.

If $\bar{M}$ is a Riemannian manifold, the Riemannian metric g of $\bar{M}$ induces a Riemannian metric (which we denote by the same letter g) on M given by

$$g(f_*X, f_*Y) = g(X, Y),$$

where f_* is the Jacobian map of f, and which indicates that the geometry of M also depends on immersion.

For every point $p \in M$, the tangent space $T_p(\bar{M})$ of $\bar{M}$ admits the following decomposition

$$T_{f(p)} \quad (\bar{M}) = T_pM \oplus T_p^{\perp} M ;$$

where $T_p(M)$ is tangent space of M at p and $T_p^{\perp} M$ is orthogonal complement of T_pM in $T_{f(p)}(\bar{M})$ consisting of all vectors normal to M.

We denote by $\nu = \bigcup_{p\in M} T_p M$, the normal bundle of M via the immersion f.

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ and $\nabla^{\perp}$ on M and in normal bundle respectively, governed by the Gauss and Wiengarten formula viz,

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) ,$$

$$\bar{\nabla}_X N = - A_N X + \nabla^{\perp}_X N ,$$

where X, Y are vector fields on M and $N \in \nu$ and h A_N are second fundamental forms related by

$$g(h(X,Y),N) = g(A_N X,Y) .$$

The curvature tensor corresponding to the connection $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are denoted by $\bar{R}$, R and $R^{\perp}$ respectively and we have the following celeberated equation due to Gauss, Caddazi and Ricci [6].

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W))) ,$$

$$[\bar{R}(X,Y)Z]^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) ,$$

$$\bar{R}(X,Y,N,N') = R(X,Y,N,N') - g([A_N, A_{N'}](X), Y) ,$$

where $[\bar{R}(X,Y)Z]^{\perp}$ denotes the normal component of $\bar{R}(X,Y)Z$

and $(\bar{\nabla}_X h)(Y,Z) = (\nabla^{\perp}_X h)(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$.

Looking into the equations, we observe that we can classify the submanifolds putting condition on h.

<u>Definition</u> 1.3.1 : A submanifold for which the second fundamental form h is zero identically is called a <u>totally geodesic submanifold</u>.

<u>Definition</u> 1.3.2 : The submanifold M is called totally umblical if its second fundamental form h satisfies

$$h(X,Y) = g(X,Y)\,H ,$$

where $H = \frac{1}{n}$ (trace h), is called <u>mean curvature</u> vector field.

<u>Definition</u> 1.3.3 : The submanifold M is called <u>minimal</u> if the mean curvature vector H vanishes identically, ie., $H = 0$. We have the result.

<u>Proposition</u> 1.3.1 : A totally umblical submanifold M in a space $\bar{M}$ of constant curvature c is also of constant curvature.

Definition 1.3.4 : Let M be a submanifold of an indefinite Riemannian manifold $\bar{M}$ (i.e. M admits an immersion $f:(M,g) \longrightarrow (\bar{M},g)$). If M is of codimension one, then M is called in Hypersurface of $\bar{M}$.

The set of basic formulas for submanifold are:

i) $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$

ii) $\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$.

where, X and Y are vector fields on M and ξ a normal vector field to M.

(i) is called Gauss's formula and (ii) is called Weigarten's formula,

In case of Hypersurface M, (ii) takes a simple form

$$\bar{\nabla}_X \xi = -A_\xi X \quad \text{since} \quad \nabla^\perp_X \xi = 0.$$

$A : TM \longrightarrow TM$ and is symmetric on each $T_p(M)$ with respect to metric on $T_p(M)$, $p \in M$, and is called Shape Operator.

Gauss equation for a hypersurface in a space form $\bar{M}(\bar{c})$ states that

$$R(U_1,U_2)U_3=\bar{c}(U_1 \wedge U_2)U_3 + \langle\xi,\xi\rangle (AU_1 \wedge AU_2)AU_3 ,$$

where R is the curvature tensor of the hypersurface, ξ a local unit normal and A the shape operator of isometric immersion.

In Geometry one comes across the problem of reducing a pair of quadratic forms with arbitrary signatures to a canonical form over the field of real numbers under the assumption that one of these form is nondegenerate and this is done in what is known as the Principal axes Theorem.

1.4 Principal Axes Theorem: Given a tensor a_{ij} in R^n with definite metric g_{ij} there exist r mutually orthogonal vectors λ^i_α $(\alpha=1,\dots,r)$ such that a_{ij} admits the representation.

$$a_{ij} = \sum_{\alpha=1}^{r} s_\alpha \lambda^i_\alpha \lambda^j_\alpha$$ where s_α are invariant and r is the rank of matrix $[a_{ij}]$.

The vectors λ^i_α occuring in above theorem, are called the Principal axes of the tensor a_{ij}.

Remark : The above theorem need not be true for an undefinite metric g_{ij}.

Let us consider two quadratic forms in R^n: $P = g_{ij}x^i x^j$ and $Q = a_{ij}x^i x^j$ with real cofficients and $|g_{ij}| \neq 0$. To determine eigen vectors i.e. principal axes of a_{ij} defined by $(a_{ij} - \lambda g_{ij})\lambda^j = 0$, one must consider the roots of the equation $|a_{ij} - \lambda_{ij}| = 0$. Roots may be complex for an indefinite g_{ij}.

For the real matrix $A = G^{-1}T = [a^i_j]$, where $G = [g_{ij}]$ and $T = [a_{kj}]$, the Hamilton-Cayley Theorem (Shirkov [31] pp. 174-9, Klein [32] pp 385-93) shows that real transformation to new coordinates x^{*i} can reduce both forms P and Q to sum of k forms, i.e., $P = \sum_1^k P_i$, $Q = \sum_1^k Q_i$ where P_i and Q_i contain only the variable x^{*j} $(j = j_1\ j_2, \ldots, j_i)$. If the matrices of the forms P_s and Q_s are denoted by G^*s and T^*s respectively, then the characteristic equation of the matrix $A^* = G_s^{*-1} \cdot T_s^*$ is $(\lambda - \lambda_\alpha)^{n_s} = 0$.

In other words, it is possible to choose new basis vectors $\overset{*}{\lambda}{}^i_\alpha = C^\beta_\alpha\ \lambda^i_\beta$ so that

$$G^* = \begin{bmatrix} G^*_{n_1} & & \\ & \ddots & \\ & & G^*_{n_k} \end{bmatrix}, \qquad T^* = \begin{bmatrix} T^*_{n_1} & & \\ & \ddots & \\ & & T^{**}_{n_k} \end{bmatrix},$$

$$A^* = \begin{bmatrix} A^*_{n_1} & & \\ & \ddots & \\ & & A^*_{n_k} \end{bmatrix},$$

where $G^*_{n_i}$, $T^*_{n_i}$ and $A^*_{n_i}$ are square matrices of order n_i.

Let us consider the case when the characterstic roots are real, using only real transformation we reduce matrices $A^*_{n_i}$ to the Jordon canonical form ([31] pp. 180-5 and [33] pp. 60-61)

$$A^*_{n_i} = \begin{bmatrix} B_{s_1} & & \\ & \ddots & \\ & & B_{s_i} \end{bmatrix}, \quad \sum_{\ell=1}^{i} S_\ell = n_i ,$$

$$Bs_\ell \quad \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix},$$

where B_s if of order S_ℓ.

It is now possible to define matrices G_{n_i} and T_{n_i}. It may be assumed without loss of generality that A_{n_i} is the leading submatrix in A and contains only two submatrices $B_{s\ell}$. This means there are two invariant vectors of order m_1 and m_2 where $m_1 + m_2 = n_1$, $m_1 < m_2$, corresponding to the characteristic equation $(\lambda - \lambda_1)^{n_1} = 0$.

For matrix A_{n_1} we obtain

$$(1.4.1) \qquad a^i_j \lambda^j_h = \lambda_1 \lambda^i_h + \Delta_h \lambda^i_{h-1} \ (h=1,\ldots,h_i, i,j=1,\ldots,n_1),$$

Where $\Delta_h = 0$ for $h = 1$, $m_1 + 1$ and $\Delta_h = 1$ otherwise

Since $\lambda^i_h = \delta^i_h$.

Then $a_{ij} = \lambda_i g_{ij} + \Delta_j g_{i,j-1}$. As the tensor g_{ij} and a_{ij} are symmetric, $\Delta_i g_{j,j-1} = \Delta_j g_{i,j-1}$. Hence

$$\left[\begin{array}{ccc|ccc} & & \alpha_1 & & & \nu_1 \\ & \iddots & \vdots & & \iddots & \vdots \\ \alpha_1 & \cdots & \alpha m_1 & \nu_1 & \cdots & \nu_{m_1} \\ \hline & & \nu_1 & & & \beta_1 \\ & \iddots & \vdots & & \iddots & \vdots \\ \nu_1 & \cdots & \nu_{m_1} & \beta_1 & \cdots & \beta_{m_2} \end{array}\right]$$

Within each elementary vector corresponding to an elementary Jordon Submatrix, transformation are possible which leave relation (1.4.1) invariant. These indlude non-singular transformation which replace the coefficients α_i, β_i for $i > 1$ and $_i$ $(i=1,\ldots,m)$ by zero (Petrove [34] pp.41-42) Since $a_{ij} = (g_{ik})\,(a^k_j)$, the matrices of the forms $g_{ij}x^ix^j$ and $a_{ij}\,x^ix^j$ can be reduced to the form

$$(g_{ij}) \quad \begin{bmatrix} G_{m_1} & & \\ & \ddots & \\ & & G_{m_k} \end{bmatrix}, \quad a_{ij} = \begin{bmatrix} T_{m_1} & & \\ & \ddots & \\ & & T_{m_k} \end{bmatrix};$$

$$G_{m_i} = \begin{bmatrix} & & l_i \\ & ⋰ & \\ l_i & & \end{bmatrix}, \qquad T_{m_i} = \begin{bmatrix} & & l_i\lambda_i & \\ & ⋰ & & l_i \\ l_i\lambda_i & l_i & ⋰ & \\ & & & \end{bmatrix},$$

where $= \pm 1$.

When the characteristic roots $_i$ are real by means of a real transformation.

1.5. Foliation and Leaves:

Definition : Let M be n-dimensional C^{∞} manifold. A codimension q, C^r-foliation of M $(o \leq q \leq n,\ o \leq r \leq \infty)$ is a family $\mathcal{F} = \{L_\alpha : \alpha \quad A\}$ consisting of arcwise connected subset of M, called leaves with the following properties:

i) $L_\alpha \cap L_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$

ii) $\bigcup_{\alpha \in A} L_\alpha = M$

iii) Every point in M has a local coordinate system (U, ψ) of class C^r such that, for each Leaf L_α, the arcwise connected components $U \cap L_\alpha$ are described by

$$x^{n-q+1} = \text{constant}, \ldots, x^n = \text{constant where } x^1, x^2, \ldots, x^n$$

denote the local coordinate in the system (U, ψ).

In particular every leaf of is an $(n-q)$ dimensional submanifold of M.

Definition : Distribution is an assignment which to every point of manifold associates a subspace of a tangent space at that point.

CHAPTER-II

EINSTEIN HYPERSURFACES OF INDEFINITE SPACE FORMS

In submanifold theory geometrically most interesting is the study of submanifolds of codimension one (called Hypersurfaces). In particular, we are interested in the hypersurfaces of indefinite Riemannian manifolds of constant curvatures. Since the Geometry of hypersurfaces mostly depend on the shape operator (the Weingartan map), in the first section of this chapter we study the possible forms of the shape operators. Most of the results assembeled in this chapter are due to Magid ([16],[17],[18]).

2.1. Shape Operators:

Let $\bar{M}^{n+1}(\bar{C})$ be the n+1 dimensional indefinite Riemannian manifold of constant curvature $\bar{C}$ and M^n be the hypersurface of $\bar{M}$. In case M is Einstein hypersurface, Fialkow [10] has classified all such hypersurfaces under the restriction that the shape operators are diagonalizable. The most general result for the possible forms of the shape operator is the following

Theorem 2.1.1 : Let $n > 2$, if $f : M^n \longrightarrow \bar{M}^{(n+1)}(\bar{C})$ is an isometric immersion of an n-dimensional indefinite Riemannian manifold in (n+1) dimensional space form of constant curvature $\bar{C}$ and if M^n is Einstein, then the shape operator A_x at each point x M is either diagonalizable or can be put into one of the following two forms

$$A_x = \begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & \pm 1 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \pm 1 \\ & & & & & 0 & 0 \end{bmatrix},$$

$$A_x = \begin{bmatrix} 0 & \beta & & & \\ -\beta & 0 & & & \\ & & \ddots & & \\ & & & 0 & \beta \\ & & & -\beta & 0 \end{bmatrix},$$

With respect to some specially chosen basis. In the last case n is even and $T_x(M^n)$ has signature $(n|2, n|2)$.

__Proof:__ If $e_1, \dots, e_n$ is an orthonromal basis of $T_x(M)$ so that $g(e_i, e_j) = \sigma_i \delta_{ij}$, $\sigma_i = \pm 1$, then Ricci tensor field S of manifold M with linear connection is defined as

$$S(X,Y) = \sum_{i=1}^{n} \sigma_i \, g(R(e_i,X)\,Y,\, e_i) \qquad (2.1.1)$$

Since M^n is Einstein hypersurface, $S(X,Y) = \rho g(X,Y)$, put $\langle \xi,\xi \rangle = \tau$, we get

$$\tau[\rho - \bar{C}(n-1)]I = [(\text{tr } A)\,A - A^2]$$

or $$\tau[\rho - \bar{C}(n-1)]I = (\text{tr } A)\,A - A^2 \qquad (2)$$

According to petrove [23] a symmetric operator in an indefinite inner product space can be put into the following form

$$A = \begin{bmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_k & & & \\ & & & C_1 & & \\ & & & & \ddots & \\ & & & & & C_m \end{bmatrix}$$

where

$$B_i = \begin{bmatrix} d_i\lambda_i & d_i & & & \\ 0 & d_i\lambda_i & & & \\ & & \ddots & & \\ & & & & d_i \\ & & & & d_i\lambda_i \end{bmatrix} \qquad \begin{matrix} d_i = \pm 1 \\ \beta_i \text{ is } s_i \times s_i \end{matrix}$$

$$C_j = \begin{bmatrix} \alpha_j & \beta_j & 1 & 0 & & & \\ -\beta_j & \alpha_j & 0 & 1 & & & \\ & \alpha_j & \beta_j & 1 & 0 & & \\ & -\beta_j & \alpha_j & 0 & 1 & & \\ & & & & \ddots & & \\ & & & & & \alpha_j & \beta_j \\ & & & & & -\beta_j & \alpha_j \end{bmatrix}$$

$\beta_j \neq 0$ and C_j is $2t_j \times 2t_j$

One computes that

$$B_i^2 = \begin{bmatrix} \lambda_i^2 & 2\lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i^2 & 2\lambda_i & 1 & 0 \cdots & 0 \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 1 \\ & & & & \ddots & 2\lambda_i \\ & & & & & \lambda_i^2 \end{bmatrix}$$

$$C_j^2 = \begin{bmatrix} \alpha_j^2-\beta_j^2 & 2\alpha_j\beta_j & 2\alpha_j & 2\beta_j & 1 & 0 & \cdots & & \\ -2\alpha_j\beta_j & \alpha_j^2\ \beta_j^2 & -2\beta_j & 2\alpha_j & 0 & 1 & 0 & \cdots & \\ 0 & 0 & -\alpha_j^2-\beta_j^2 & 2\alpha_j\beta_j & 2\alpha_j & 2\beta_j & 1 & 0 & \cdots \\ & & & \ddots & & & \ddots & & \\ & & & & & \ddots & & & \\ & & & & & & & \cdots & \alpha_j^2-\beta_j^2 \end{bmatrix}$$

Letting $k = \tau(\rho - \bar{C}(n-1))$ we must have

$$kI = (\operatorname{tr} A)\, A - A^2$$

It is clear from the form of B_j^2 C_j^2 that $s_j \le 2$ $t_j \le 1$ so that A has blocks of the form

$$[\mu_i] \quad \text{or} \quad \begin{bmatrix} d_j\lambda_j & d_j \\ 0 & d_j\lambda_j \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix},$$

with squares $[\mu_i^2]$ or $\begin{bmatrix} \lambda_j^2 & 2\lambda_j \\ 0 & \lambda_j^2 \end{bmatrix}$ or $\begin{bmatrix} \alpha_k^2-\beta_k^2 & 2\alpha_k\beta_k \\ -2\alpha_k\beta_k & \alpha_k^2-\beta_k^2 \end{bmatrix}$.

By change of basis $(\ell, \hat{\ell}) \longrightarrow (-\ell, \hat{\ell})$ we can have blocks of the form

$$[\mu_i] \quad \text{or} \quad \begin{bmatrix} \lambda_j & 1 \\ 0 & \lambda_j \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}$$

With trace $A = S$, the equation $SA - SA^2 = kI$ yields

$$S - 2\lambda_j = 0, \qquad S\beta_k - 2\alpha_k\beta_k = 0,$$

$$S\mu_i - \mu_i^2 = k, \quad S\lambda_j - \lambda_j^2 = k, \quad S\alpha_k - \alpha_k^2 + \beta_k^2 = k.$$

If there are any blocks with α's and β's, $\beta_i \neq 0$ so that we have $S|2 = \lambda_j$, $S|2 = \alpha_k$, for each j & k. Thus all λ_j's & α_k's are equal. It is then clear that all β_k's are equal. The equations became

(3) $S - 2\lambda = 0$, $S - 2\alpha = 0$,

(4) $S\mu_i - \mu_i^2 = k$, $S\lambda - \lambda^2 = k$, $S\alpha - \alpha^2 + \beta^2 = k$.

Subsituting (3) in (4) we have

$$S\mu_i - \mu_i^2 = k, \ \lambda^2 = k, \ \alpha^2 + \beta^2 = k .$$

Since $\lambda = \alpha$ and $\beta \neq 0$ there can be blocks with α's or blocks with λ's but not both. In either case we have

$$\mu_i = \frac{1}{2}\left(S \pm \sqrt{S^2 - 4k^2}\right) ,$$

if $k = \lambda^2$, $\mu_i = S|2$. If $k = \alpha^2 + \beta^2$, $S^2 - 4k^2 < 0$ and there are no μ_i's.

If there is block with a λ , then $\lambda = S|2$ and $\mu_i = S|2$ for each i. If p is the number of μ's which appear in A and $2q$ the number of 's,

$$S = p\mu + 2q\lambda = p(S|2) + 2q(S|2)$$

Thus $S(1-p|2-q) = 0$, but $p + 2q \geq 3$ so $s = 0$. One possibility for A then is

$$\begin{bmatrix} 0 & & & & \\ & 0 & +1 & & \\ & 0 & 0 & & \\ & & & 0 & \pm 1 \\ & & & 0 & 0 \end{bmatrix}$$

If there is block with a β, there are no other types of blocks. Since $\alpha = \ |2$ we again see that $\ = o$ and

$$A = \begin{bmatrix} 0 & \beta & & \\ -\beta & 0 & & \\ & & 0 & \beta \\ & & -\beta & 0 \end{bmatrix}.$$

Q.E.D.

We give the following examples to demonstrate the shape operators which are not diagenalizable.

Example 1 : Consider $R_n^{2n} \longrightarrow R_n^{2n+1}$ with the following immersion.

$$(x_1, \ldots, x_{2n-1}, x_{2n}) \longrightarrow (x_1 + x_2, x_3 + x_4 \ldots, x_{2n-1} + x_{2n}, \\ x_1 - x_2, \ldots, x_{2n-1} - x_{2n}, x_2^2 + x_4^2 \ldots x_{2n}^2),$$

with the standard inner product $(-,\dots,-,+,\dots,+)$ with n negative signs. The shape operator is

$$\begin{bmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix}$$

at each point.

<u>Example</u> 2 : $CS^n(1) = \{(Z_1,\dots,Z_{n+1}) \in C^{n+1} : Z_1^2 + \dots + Z_{n+1}^2 = 1\}$ in S^{2n+1}_{n+1} has shape operator

$$\begin{bmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

at each point

As a direct consequence of theorem 2.1.1 we have the following corollary:

<u>Corollary</u> 2.1.1: If $F : M^{2n} \longrightarrow \bar{M}^{2n+1}(\bar{C})$ is an isometric and immersion of an Einstein manifold and if A_X is not diagonalizable at each point. Then $A^2 = 0$ everywhere or $A^2 = -b^2 I$

everywhere, for b_i nonzero constant.

As a consequence of Theorem 2.1.1 we observe that for indefinite Einstein Hypersurfaces either shape operator is diagonalizable or has the form $A^2 = 0$, $A^2 = -b^2I$. A Fialkow has completely classified the hypersurfaces with diagonalizable shape operator.

In the following section we study indefinite Einstein hypersurfaces in which either $A^2 = 0$ or $A^2 = -b^2I$.

2.2. Indefinite Hypersurfaces with $A^2 = 0$

For even dimensional Indefinite Einstein hypersurfaces we have the following result.

Theorem 2.2.1 : If $f : M^{2n} \longrightarrow \bar{M}^{2n+1}(\bar{C})$ is an isometric immersion of an Einstein manifold with $A_x^2 = 0$, rank $A_x = n$ for all $x \in M^{2n}$, then Ker A is a smooth integrable, totally geodesic and totally degenerate n-dimensional distribution on M.

Proof: Choose $U_1, \ldots, U_n$ at p such that $AU_j \neq 0$ and $U_1, \ldots, U_n$ are linearly independent. Then in a neighbourhood of p, $AU_j \neq 0$, since $AAU_j = 0$, $AU_1, \ldots, AU_n$ form a basis for Ker A in a neighbourhood of p and Ker A is smooth n-dimensional distribution.

If $X, Y \in \mathrm{Ker}\, A$, we have by Codazzi equation that

$$A(\nabla_X Y) - \nabla_X(AY) = A(\nabla_Y X) - \nabla_Y(AX),$$

$$A(\nabla_X Y) - A(\nabla_Y X) = 0, \quad A\,[X,Y] = 0,$$

and $\mathrm{Ker}\, A$ is integrable.

It is easy to see that $A^2 = 0$, $\mathrm{rank}\, A = n$, implies that $\mathrm{Ker}\, A = \mathrm{im}\, A$.

If $U, V \in T_x(M)$, $\langle AU, AV\rangle = \langle A^2U, V\rangle = 0$ so that $\mathrm{Ker}\, A$ is totally degenerate, i.e., has no metric.

Finally if $X, Y \in \mathrm{Ker}\, A$, then

$\nabla_X Y \in \mathrm{Ker}\, A$, $\langle Y, AU\rangle = 0$ so

$$X\langle Y, AU\rangle = \langle \nabla_X Y, AU\rangle + \langle Y, \nabla_X(AU)\rangle$$

$$= \langle \nabla_X Y, AU\rangle + \langle Y, \nabla_U(AX)\rangle + \langle Y, A\,[U,X]\rangle$$

$$= \langle \nabla_X Y, AU\rangle .$$

Since $AX = AY = 0$, thus $A(\nabla_X Y) \perp U$ for all U and $A(\nabla_X Y) = 0$.

Q.E.D.

We know that for a symmetric operator A on a vector space V with nondegenerate inner product $\langle\ ,\ \rangle$, if $A^2 = 0$, we can find a basis $(\bar{e}_1, e_1, \ldots, \bar{e}_n, e_n, E_1, \ldots, E_p)$ of V with respect to which

$$\begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 1 & \\ & & 0 & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

Here e_i, $\bar{e}_j$ are light like, $\langle e_i, \bar{e}_j \rangle = -\delta_{ij}$ and $\langle E_k, E \rangle = \pm\delta_{k\ell}$ and all other inner products are zero.

Furthermore it is also known that if ratio of the rank of A to the dimension of V is as large as possible then $p = 0$ and the basis satisfies $A\bar{e}_i = 0$ and $A\bar{e}_i = \bar{e}_i$. In such case is even dimensional and has signature (n,n), then we have the following lemma:

<u>Lemma</u> 2.2.1 : If $f : M^{2n}_n \longrightarrow N^{2n+1}_{(C)}$ is an isometric immersion with $A^2 = 0$ and $\mathcal{N}(A) = n$, then there are vector fields

$\bar{e}_1,\ldots,\bar{e}_n$ defined in a neighbourhood of any point of M such that $(\bar{e}_i,\bar{e}_j) = 0 = (A\bar{e}_i, A\bar{e}_j)$ & $(A\bar{e}_i, e_j) = -\delta_{ij}$.

<u>Theorem</u> 2.2.2 : If $f : M_n^{2n} \longrightarrow N^{2n+1}_{(C)}$ is an isometric immersion of M_n^{2n} into a space form of constant curvature C with $A^2 = 0$ and $\pi(A) = n$ then the Kernel of A is an integrable, totally isotropic and parallel n-dimensional distribution on M.

<u>Proof</u>: From Theorem 2.2.1 it follows that $K(A)$ is integrable, totally geodesic and totally isotropic (namely totally degenerate). A totally geodesic distribution S is one where

$$\nabla_X Y \in S \quad \text{if} \quad X,Y \in S$$

To prove that Kernel A is parallel we must show that $\nabla_U X \in \text{Ker } A$ if $X \in \text{Ker } A$ and $U \in TM$ or equivalently, that

$$A(\nabla_U X) = 0 \quad \text{if} \quad AX = 0.$$

In order to do this, let $x \in M$ and choose vector fields in a neighbourhood of x, $(e_1,\ldots,e_n, Ae_1,\ldots,Ae_n)$ as in Lemma 2.2.1.

Consider Codazzi's equation with L_i and L_j, $1\leq i, j\leq n$.

$$\nabla_{e_i}(Ae_j) - A(\nabla_{e_i} e_j) = \nabla_{e_j}(Ae_i) - A(\nabla_{e_j} e_i)$$

The inner product of both sides of this equation with Ae_k gives

$$(\nabla_{e_i} Ae_j, Ae_k) = (\nabla_{e_j} Ae_i, Ae_k) \qquad (2.2.2)$$

Since $A^2 = 0$, denoting Ae_j by $e_{j'}$, $j = 1,\ldots,n$ and defining Γ^D_{BC} the Christoffel symbols, as usual we have

$$\nabla_{e_i} e_{j'} = \sum_{k=1}^{n} \Gamma^k_{ij'} e_k + \Gamma^{k'}_{ij'} e_{k'}$$

then (2.2.2) becomes $\Gamma^k_{ij'} = \Gamma^k_{ji'}$, $1 \le i, j, k \le n$. (2.2.2')

Because the connection in M is metric

$$e_i(Ae_j, Ae_k) = 0 = (\nabla_{e_i} Ae_j, Ae_k)$$

$$+ (Ae_j, \nabla_{e_i} Ae_k)$$

so that

$$\Gamma^k_{ij'} + \Gamma^j_{ik'} = 0, \quad 1 \le i, j, k \le n \qquad (2.2.2'')$$

Combining (2.2.2') and (2.2.2") we see that $\Gamma^k_{ij'} = 0 \ \forall$

$1 \le i, j, k \le n$.

In fact

$$\Gamma^k_{ij'} = \Gamma^k_{ji'} = -\Gamma^i_{jk'} = -\Gamma^i_{kj'} = \Gamma^j_{ki'} = \Gamma^j_{ik'} = --\Gamma^k_{ij'}.$$

The fact that the Kernel of A is totally geodesic gives $\Gamma^k_{ij'} = 0$. Thus $\Gamma^k_{Bj'} = 0$ for $B = 1,\ldots,n,1',\ldots,n'$; $1 \leq j,k \leq n$. This means that Kernel of A is parallel.

Q.E.D.

As an immediate consequence of above theorem we have the following:

<u>Corollary</u> 2.2.1 : Let $n > 1$. If $f : M^{2n}_n \longrightarrow N^{2n+1}_{(C)}$ is an isometric immersion of M^{2n}_n into space form of constant curvature C with $A^2 = 0$ and rank $A = n$, then $C = 0$.

<u>Remark</u> : In the above corollary the restriction $n > 1$ cannot be removed because Graves and Nomizo [12] constructed an example of Lorentz surface M^2_1 isometrically immersed in $^3_1$ with A satisfying $A^2 = 0$ and rank $(A) = n$.

In Eucledian spaces with Indefinite inner product the converse of Theorem 2.2.2 is also true as exhibited in the following;

<u>Theorem</u> 2.2.3 : Let $f : M_n^{2n} \longrightarrow R_n^{2n+1}$ be an isometric immersion with rank $A = n$. Then Kernel A is an integrable, totally isotropic, parallel distribution on M_n^{2n} iff $A^2 = 0$.

In the end of this section we give some example of Indefinite Einstein hypersurfaces with $A^2 = 0$ and $\eta(A) = n$.

<u>Example</u> 3. B-scroll over a null curve in R_1^3 (Lorentz 3-space with signature $(-, +\, +)$.) [13].

Consider a null curve $x(s)$ in R_1^3 so that $\left(\frac{dx(s)}{ds}, \frac{dx(s)}{ds}\right) = 0$. Consider cartan framed null curve $\{A(s), B(s), C(s)\}$ such that

$$A(s),\ B(s) \text{ are null } ; \ (C(s), C(s)) = 1,$$

$(A(s), B(s)) = -1$ and all other inner products are zero. along $x(s)$, and Frenet equation of the derivatives of $A(s)$, $B(s)$, $C(s)$ along $x(s)$ have the form

$$\frac{dx(s)}{ds} = A(s) \ ,$$

$$\frac{dA(s)}{ds} = k_2(s)\, C(s) \ ,$$

$$\frac{dB(s)}{ds} = k_3(s)\, C(s) \ ,$$

$$\frac{dC(s)}{ds} = k_3(s)\, A(s) + k_2(s)\, B(s) \ .$$

The surface $f(U,s) = x(s) + UB(s)$ is called B-Scroll over the null curve $x(s)$.

It is Lorentz and is flat iff $K_3(s) = 0$. In this case

$$A = \begin{bmatrix} 0 & -k_2(s) \\ 0 & 0 \end{bmatrix}$$

With respect to $\left\{\frac{\partial}{du}, \frac{\partial}{\partial s}\right\}$, where unit normal $\xi(U,s) = C(s)$, $\nabla A = 0$ iff $k_2(s)$ is constant. If $k_2 \equiv 1$, the surface is given by

$$x(s) + UB(s) = \left(\frac{s^3}{6\sqrt{2}} + \frac{s}{\sqrt{2}}, \frac{s^3}{6\sqrt{2}} - \frac{s}{\sqrt{2}}, \frac{s^2}{2}\right) + U\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right).$$

Graves calls this the B-scroll over null cubic.

3. Indefinite Einstein Hypersurface $A^2 = -b^2 Id$

In the previous sections we have discussed the Einstein hypersurfaces in indefinite space form whose shape operators are nilpotent. In this last section we consider the remaining class of Einstein hypersurfaces of indefinite space form, namely those whose shape operators satisfy $A^2 = -b^2 Id$ where b is real constant.

To start with we define the complex spheres $C S^n(\rho)$ by

$C^n(\ ) = \{(z_1,\dots,z_{n+1}) \in C^{n+1} / z_1^2 + z_2^2 + \dots z_{n+1}^2 = \ \}$

Using halonomy theorem of Ambrose and singer [12] it is not difficult to see $C^n = \frac{SO(n+1,C)}{SO(n,c)}$ and that C^n is an irreducible symmetric space.

Let M be the hypersurface of indefinite space form $\bar{M}^{n+1}_{(C)}$. The tangent bundle TM of M as well as g, and A can be complexified to yield TM^C, g^C, ∇^C and A^C, for example

$$\nabla^C_{X+iY}(U+iV) = \nabla_X U - \nabla_Y V + i(\nabla_Y U + \nabla_X V),$$

We have the following proposition whose proof is immediate.

<u>Proposition</u> 2.3.1 : If Z, W and W' are in TM^C then g^C, ∇^C and A^C satisfy

i) $\nabla^C_Z W - \nabla^C_W Z = [Z,W]$,

ii) $Zg^C(W,W') = g^C(\nabla^C_Z W,W') + g^C(W,\nabla^C_Z W')$,

iii) $\nabla^C_Z (A^C W) - A^C(\nabla^C_Z W) = \nabla^C_W (A^C Z) - A^C(\nabla^C_W Z)$,

so that ∇^C is torsion free and metric and A^C satisfies the codazzi Equation.

<u>Proposition</u> 2.3.2 : M^{2n}_n be isometrically immersed as a hypersurface in an indefinite space form, if $A^2 = -b^2 Id$ and

$b \neq o$, then A is parallel.

<u>Proof</u>: From Proposition 2.3.2 it follows that, if $A^2 = -b^2 Id$, $b \neq o$, then we can find a basis of $T\ (M)$ at every point $p\ \ M$ with respect to which

$$A = \begin{bmatrix} 0 & b & & \\ -b & 0 & & \\ & & & \\ & & 0 & b \\ & & -b & 0 \end{bmatrix}$$

in order to show that A is parallel we need a smooth frame in the neighbourhood of $p \in M$ with respect to which A has above form.

A has exactly two eigen value $\pm$ ib, they show that we can find a smooth basis $\{Z_1, \ldots, Z_n\}$ of ib eigen space $T_{ib} = \{Z : AZ = ibZ\}$ in the nbd. of $p \in M$.

Writing $Z_j = X_j + iY_j$, $1 \leq i, j \leq n$, we see that $AZ_j = ibZ_j$ means

$$AX_j = -bY_j,$$

$$AY_j = bX_j,$$

Again following [24] we see that

$$\nabla_{Z_j} Z_k \in T_{i_b} \quad \text{and} \quad \nabla_{\bar{Z}_j} Z_k \in T_{i_b} ,$$

$$1 \leq j, \ k \leq n$$

Since there are two principal curvatures and Rayon [24] has shown that these are not only integrable but are also parallel. Thus comparing real and imaginary parts of the equation

$$A(\nabla_{Z_j} Z_k) = ib \nabla_{Z_j} Z_k$$

$$= \nabla_{Z_j} ib Z_k ,$$

$$A(\nabla_{Z_j} Z_k) = \nabla_{Z_j} A Z_k ,$$

it direclty follows that $\nabla A = 0$, i.e., A is parallel.

Q.E.D.

On the hypersurface M^{2n} we define an almost complex structure J by setting $J = (\frac{1}{b}) A$, then it is easy to verify $g(JX, JY) = -g(X, Y)$

Now we have the following Lemma

__Lemma__ 2.3.1 : An indefinite Riemannian manifold with a complex structure J, satisfying $\nabla J = 0$ and $g(JX, JY) = -g(X, Y$

has $R(X,JX) = 0 \quad \forall \quad X \in \mathfrak{X}(M)$.

<u>**Proof:**</u> First it is noted that

$$J.R(X,Y) = R(X,Y).J \quad \text{because} \quad J(\nabla_X Y) = \nabla_X (J Y),$$

it is also true that

$$R(X,Y) = -R(JX, JY),$$

In fact

$$g(R(X,Y)V,U) = g(R(U,V)Y,X)$$
$$= -g(JR(U,V)Y,JX)$$
$$= -g(U,V)JY, JX)$$
$$= -g(R(JX,JY)V,U)$$

$\forall \; U,V,X,Y$ in $\mathfrak{X}(M)$ this gives

$$R(X,Y) = -R(JX\ JY),$$
$$R(X,JX) = -R(JX, J^2X)$$
$$= R(JX\ X)$$
$$= -R(X, JX),$$

$2R(X\ JX) = 0 \implies R(X,JX) = 0 \quad \forall X \in \mathfrak{X}(M)$.

Q.E.D.

The main theorem of this section is

<u>Theorem</u> 2.2.1 : Let M^{2n}, $n > 1$ be simply connected hypersurface of an indefinite space form of constant curvature C. If $A^2 = -b^2 I$, b then $C = b^2$ or $-b^2$ and $M^{2n} = C^{n}(\frac{1}{b})$ or $C^{n}(\frac{1}{b})$.

<u>Proof</u>: Let ξ be a unit normal to M and R be the curvature tensor of M, the Gaun equation is

$$R(X,Y) = C(X \wedge Y) + \langle \xi, \xi \rangle (AX \wedge AY)$$

$$= C(X \wedge Y) + \langle \xi, \xi \rangle b^2 (JX \wedge JY)$$

$$\text{as } J = (\tfrac{1}{b}) A ,$$

Where $X \wedge Y : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ and is defined by

$(X \wedge Y)Z = g(Y,Z)X - g(X\ Z)\ Y.$

By Lemma 2.3.1

$$R(X, JX) = 0 ,$$

$$\Longrightarrow \quad C(X \wedge JX) + \langle \xi, \xi \rangle b^2 (JX \wedge J^2X) = 0 ,$$

$$C(X \wedge JX) - \langle \xi, \xi \rangle b^2 (JX \wedge X) = 0 ,$$

$$C(X \wedge JX) + \langle \xi, \xi \rangle b^2 (X \wedge JX) = 0 ,$$

$$(C + \langle \xi, \xi \rangle b^2) (X \wedge JX) = 0$$

Therefore $C = - \langle \xi, \xi \rangle b^2 \neq 0$, if $\langle \xi, \xi \rangle = -1$, $C = b^2$,

$\langle \xi, \xi \rangle = 1$, $C = -b^2$.

We see then that M_n^{2n} isometrically immersed in $S_{n+1}^{2n+1}(1|b)$ has $R(X,JX) = b^2(X \wedge Y) - b^2(JX \wedge JY)$ and when M_n^{2n} isometrically immersed in $H_1^{2n+1}(1/b)$ has $R(X,JX) = -b^2(X \wedge Y) - b^2(JX \wedge JY)$. Furthermore, since TS^n is homeomorphic to CS^n, the fundamental group of CS^n is given by $\pi_1(CS^n) = 0$ for $n > 1$. i.e., CS^n is simply connected. Hence, by a result (cf. [26], ch.9, p. 170) M_n^{2n} is isometric to $CS^n(1/b)$ or $CS^n(i/b)$.

Q.E.D.

CHAPTER-III

INDEFINITE HYPERSURFACE OF LORENTZ SPACE FORM

After having studied the Einstein hypersurfaces of Indefinite space form, the next step would be to study general hypersurfaces of indefinite space forms. However, as a step between the two stipulate an additional condition on the ambient manifold to be a Lorentz space form. Thus in this chapter we assemble some results on indefinite hypersurfaces of Lorentz space form.

The basic results are due to Magid (cf. [19])

3.1. Structure of the Shape Operator:

In this section we state some basic results about indefinite Riemannian submanifolds of indefinite Riemannian manifold. These include an indefinite version of the result of Allendoerfer and Erbacher (cf. [7]) for reducing the co-dimension of an isometric immersion with parallel second fundamental form ($\overset{\perp}{\nabla} h = o$) and an improved version of Petrov's Canonical form for symmetric transformation of Lorentz space.

Lemma 3.1.1 : If $f : M \longrightarrow \bar{M}$ is an isometric immersion of one indefinite Riemannian manifold into another with parallel second fundamental form, then

1) The mean curvature vector is parallel i.e. $\overset{\perp}{\nabla} \eta = o$.

ii) The first normal space $N^{1}(x)$, $x \in M$ are $\nabla^{\perp}$ parallel.

$$N^{1}(x) = \{N^{o}(x)\}^{\perp} \text{ where } N_{o}(x) = \{\xi \in N(x) : A_{\xi} = o\}$$

The following Theorem gives the reduction in co-dimension of an isometric immersion with parallel second fundamental form.

Theorem 3.1.1 : Let $f : M_{i}^{n} \longrightarrow R_{j}^{m}$ be an isometric immersion of an indefinite Riemannian manifold with signature (i, n-i) into R_{j}^{m}. If the first normal spaces are parallel, then there exist a complete (n+k) dimensional totally geodesic submanifold M^{*} of R_{j}^{m}
(where $n = \dim M$ and $k = \dim N^{1}$) such that $f(M) \subset M^{*}$.

The following Theorem is essentially a version of Moore's Theorem (cf. [21]) on product of isometric immersion in indefinite case, which follows directly from the last theorem.

Theorem 3.1.2 : Let M be an indefinite Riemannian product $M_1 \times M_2$. If $f : M \longrightarrow R_{j}^{n}$ be an isometric immersion with $h(X,Y) = o$ whenever $x \in Tx(M_1)$ and $y \in Tx(M_2)$ then

$$f_1 \times f_2 : M_1 \times M_2 \longrightarrow R_{j_1}^{n_1} \times R_{j_2}^{n_2}$$ and each f_i is an

isometric immersion.

Now we state the following algebraic result which is important for analysing the structure of the shape operators.

<u>Theorem</u> 3.1.3 : Let V be a real n-dimensional vector space equipped with a Lorentzian metric and let A be linear transformation of V which is symmetric with respect to this metric. Then A can be put into one of the following four forms with respect to bases whose inner products are given by G.

$$A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad G = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} \lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} \lambda & 0 & 1 & & & \\ 0 & \lambda & 0 & & & \\ 0 & 1 & \lambda & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-3} \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} \alpha & \beta & & & \\ -\beta & \alpha & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{bmatrix}, \quad G = \begin{bmatrix} +1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

These will be referred to as the case where there are

i) Simple real eigen values

ii) non simple eigen value of multiple 2

iii) non simple eigen value of multiple 3

or

iv) is a complex eigen value.

Now we state the following proposition to see that what form Lorentzian symmetric matrix takes if it commutes with one of the four standard Lorentzian symmetric matrices.

<u>Proposition</u> 3.1.1 : Let A and B be square matrices which are symmetric with respect to a Lorentzian inner product. If with respect to pseudo orthonormal basis

$$A = \begin{bmatrix} \lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \lambda I_{k_0} & & \\ & & & \lambda_1 I_{k_1} & \\ & & & & \ddots \ \lambda_s I_{k_s} \end{bmatrix}$$

and B commutes with A, then

$$B = \left[\begin{array}{ccccc|cc} \mu & b & c_1 & \cdots & c_{k_0} & & \\ 0 & \mu & 0 & & & & \\ \vdots & c_1 & & d^0_{ij} & & & \\ \vdots & \vdots & & & & & \\ 0 & c_{k_0} & & & & & \\ \hline & & & & & \boxed{d^1_{ij}} & \\ & & & & & & \ddots \ d^s_{ij} \end{array}\right]$$

where d^{h}_{ij} is $k_h \times k_h$

<u>Proposition</u> 3.1.2 : Let A and B be square matrices which are symmetric with respect to a Lorentzian inner product. If with respect to pseudo orthonormal basis

$$\begin{bmatrix} \lambda & 0 & 1 & & & & \\ 0 & \lambda & 0 & & & & \\ 0 & 1 & \lambda & & & & \\ & & & \lambda I_{k_0} & & & \\ & & & & \lambda_1 I_{k_1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_s I_{k_s} \end{bmatrix}$$

and B commutes with A, then

$$\left[\begin{array}{cccccc|ccc}
\mu & b & c_1 & c_2 & \cdots & c_{k_0+1} & 0 & \cdots & 0 \\
0 & \mu & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & c_1 & \mu & & & & & & \\
0 & c_2 & & & & & & & \\
\vdots & \vdots & & & d^{0}_{ij} & & & & \\
0 & c_{k_0+1} & & & & & & & \\
\hline
0 & 0 & & & & & d^{1}_{ij} & & \\
\vdots & \vdots & & & & & & \ddots & \\
0 & 0 & & & & & & & d^{s}_{ij}
\end{array}\right]$$

where d^{r}_{ij} is $k_r \times k_r$.

<u>Proposition</u> 3.1.3 : Let A and B be square matrices which are symmetric with respect to Lorentz inner product. If with respect to an orthonormal basis

$$A = \begin{bmatrix}
\alpha & \beta & & & & \\
-\beta & \alpha & & & & \\
& & \alpha I_{k_0} & & & \\
& & & \lambda_1 I_{k_1} & & \\
& & & & \ddots & \\
& & & & & \lambda_s I_{k_s}
\end{bmatrix} \qquad \beta \neq 0$$

and B commutes with A, then

$$B = \begin{bmatrix} \gamma & \delta & & & \\ -\delta & \gamma & & & \\ & & d^{o}_{ij} & & \\ & & & \ddots & \\ & & & & d^{s}_{ij} \end{bmatrix}$$

where d^{r}_{ij} is $kr \times kr$

After having stated some algebraic results above, we are now in a position to state the main classification theorem concerning totally umbilical isometric immersion

$$M^n_j \longrightarrow R^m_k \ , \quad M^n_i \longrightarrow R^{n+k+s}_{j,s} \quad \text{with}$$

parallel second fundamental form and some special cases thereof.

Definition 3.1.1 : R^p_{ij} denotes p-demensional affine space with the metric (.) whose canonical form is

$$\begin{bmatrix} -I_i & & \\ & I_{p-i-j} & \\ & & 0_j \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix and O_j is $j \times j$ zero matrix. The metric is nondegenerate iff $j = o$ in which case we write R^p_i .

<u>Theorem</u> 3.1.4 : *If $f : M^n_i \longrightarrow R^m_k$ is an isometric immersion and there is globally defined normal vector field N such that*

i) N is everywhere non-zero

ii) $\nabla^{\perp} N = o$

iii) $A_N = \lambda\, Id, \quad \neq o$

then $f(M^n_j)$ is contained inside

$$S^{m-1}_k \left(\frac{\sqrt{\langle N,N \rangle}}{\lambda} \right) , \text{ if } \quad \langle N, N \rangle > 0 \ ,$$

$$H^{m-1}_{k-1} \left(\frac{-\sqrt{-\langle N,N \rangle}}{\lambda} \right) , \text{ if } \quad \langle N,N \rangle < 0 \ ,$$

and $\quad LC^{m-1}_{k-1,1}$, if $\langle N,N \rangle = 0$. .

As a direct consequence of above theorem we have

<u>Corollary</u> 3.1.1 : If η is the mean curvature vector, then $f\ (M^n_j)$ is immersed minimally in S^{m-1}_k or H^{m-1}_{k-1}

§ 3.2. Submanifold with Parallel second fundamental form:

In the following section we study the immersion with parallel second fundamental form in endefinite Eucledian space.

Lemma 3.2.1 : Suppose $f : M_i^n \longrightarrow R_{j,s}^{n+k+s}$ is an isometric immersion. If $p : R_{j,s}^{n+k+s} \longrightarrow R_j^{n+k}$ is projection onto the first $n+k$ co-ordinates. Then $p \circ f : M_i^n \longrightarrow R_j^{n+k}$ is an isometric immersion.

Proof : Choose x_o in M and let U be a co-ordinate neighborhood of x_o with co-ordinates $(x^1, x^2, \dots, x^n)$.

Let $f(x^1, x^2, \dots, x^n) = (f^1, f^2, \dots, f^{n+k}, \ell^1, \dots, \ell^s)$

where the metric vanishes in the last s co-ordinates, clearly

$$g\left(\frac{\partial}{\partial x^t}, \frac{\partial}{\partial x^m}\right) = \left(f_* \frac{\partial}{\partial x^t}, f_* \frac{\partial}{\partial x^m}\right)$$ (because

f is isometric immersion)

$$= \left((p \circ f)_* \frac{\partial}{\partial x^t} \ (p \circ f)_* \frac{\partial}{\partial x^m} \right).$$

Q.E.D.

<u>Lemma</u> 3.2.2 : Let $f : M_t^n \longrightarrow R_{r,s}^{n+m+s}$ be an isometric immersion with parallel second fundamental form.

If $f(x^1,\dots,x^n) = (f^1,\dots,f^{n+m}, \ell^1,\dots, \ell^s)$ locally, then

$$\frac{\partial^3 \ell^p}{\partial x^i \, \partial x^j \, \partial x^k} = \sum_{u=1}^{n} \Gamma^u_{ji} \left(\frac{\partial^2 \ell^p}{\partial x^u \, \partial x^k} \right)$$

$$+ \sum_{v=1}^{n} \Gamma^v_{jk} \frac{\partial^2 p}{\partial x^v \, \partial x^i} .$$

<u>Proof</u>: If f has parallel second fundamental form, then

$$\nabla^{\perp}_Z h(X,Y) = h(\nabla_Z X,Y) + h(X,\nabla_Z Y)$$

If p is projection onto last s co-ordinates, then

$$\nabla^{\perp}_Z \, p \circ h(X,Y) = p \circ h(\nabla_Z X,Y) + p \circ h(X,\nabla_Z Y) .$$

Replacing X,Y,Z by $\frac{\partial}{\partial x^j}$, $\frac{\partial}{\partial x^k}$ and $\frac{\partial}{\partial x^i}$ respectively, yields the lemma, since it is well known

$$\frac{\partial^2 f}{\partial x^j \, \partial x^k} = D_{\partial/\partial x^j} \, f_*\left(\frac{\partial}{\partial x^k} \right), \Big(D \text{ is flat connection on } R_j^{n+k} \Big)$$

$$= f_* \left(\nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^k} \right) + h\left(\frac{\partial}{\partial x^j} , \frac{\partial}{\partial x^k} \right)$$

Q.E.D.

We have the following classification Theorem which is more general than Theorem 3.1.4.

<u>Theorem</u> 3.2.1 : Let $F : M_k^n \longrightarrow R_j^m$ be a complete umblical immersion. Assume the mean curvature vector $\eta \neq o$ so that f is not totally geodesic.

i) If $\langle \eta, \eta \rangle > o$ then $f(M) \subset S_k^n \subset R_k^{n+1}$

ii) If $\langle \eta, \eta \rangle < o$ then $f(m) \subset H_k^n \subset R_{k+1}^{n+1}$

iii) If $\langle \eta, \eta \rangle = o$ then $M = R_k^n$ and $f(M) \subset R_{k,1}^{n+1}$

as an umblic.

<u>Proof</u>: i) If $\langle \eta, \eta \rangle \neq o$, then by Theorem 3.1.1 there is an R_k^{n+1} or R_{k+1}^{n+1} into which M_k^n is isometrically immersed depending on the sign of $\langle \eta, \eta \rangle$. This mapping is umbilical and by Corollary 3.1.1

$$f(M) \subset S_k^n \subset R_k^{n+1}$$

or

$$f(M) \subset H_k^n \subset R_{k+1}^{n+1}$$

ii) If $\langle \eta, \eta \rangle = o$, the image of M_k^n is in $R_{k,1}^{n+1}$. By Lemma 3.2.1, $f : M_k^n \longrightarrow R_k^n$ is an isometric immersion so that $M_k^n = R_k^n$.

Lemma 3.2.2 allow us to determine all such that

$$f(x^1,\ldots,x^n) = (x^1,\ldots,x^n, \quad (x^1,\ldots,x^n))$$

we have

$$\frac{\partial^2 f}{\partial x^i\, \partial x^j} = D_{\frac{\partial}{\partial x^i}} \; f_* \left(\frac{\partial}{\partial x^j} \right)$$

$$= f_*\left(\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j}\right) + h\left(\frac{\partial}{\partial x^i} , \frac{\partial}{\partial x^j} \right) .$$

Now

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial x^1}{\partial x^i},\ldots, \frac{\partial x^i}{\partial x^i},\ldots, \frac{\partial x^n}{\partial x^i} , \frac{\partial \ell\,(x^1,\ldots,x^n)}{\partial x^i}\right)$$

$$= (0,\ldots,0,\ldots,1,\ldots 0, \; \frac{\partial \ell\,(x^1,\ldots,x^n)}{\partial x^i}) .$$

$$\frac{\partial^2 f}{\partial x^i\, \partial x^j} = (0,\ldots,0,\ldots,0, \quad \frac{\partial^2 \ell\,(x^1,\ldots,x^n)}{\partial x^i\, \partial x^j})$$

$$= (0,\ldots,0,\ldots,\ell_{ij})$$

Thus

$$(0,\ldots,0, \ell_{ij}) = f_*\left(\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j}\right) + h\left(\frac{\partial}{\partial x^i} , \frac{\partial}{\partial x^j} \right)$$

$$\Longrightarrow \quad h\left(\frac{\partial}{\partial x^i} , \frac{\partial}{\partial x^j} \right) = (0,\ldots,0,\ldots,\ell_{ij})$$

$$\left(\text{since } \; f_*\left(\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j}\right) = 0\right).$$

since f is umbilical, then

$$h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\beta$$

$$\Longrightarrow \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\beta = (0,\ldots,0,\ell_{ij}) ,$$

it follows easily that ℓ is of the desired form.

Q.E.D.

We now prove some Lemmas.

<u>Lemma</u> 3.2.3 : Let $f : R^n_j \longrightarrow R^m_k$ be an isometric immersion with parallel second fundamental form. If $(x^1,\ldots,x^n)$ is global co-ordinate system with $\nabla_{\partial i}\, \partial_j = 0$, $i,j = 1,\ldots,n$, then $h(\partial_i, \partial_j)$ is global parallel normal field.

<u>Proof</u> : By definition

$$\nabla_Z h(X,Y) = \overset{\perp}{\nabla}_Z h(X,Y) - h(\nabla_Z X,Y) - h(X,\nabla_Z Y) = 0 ,$$

replacing X,Y,Z by $\partial_i, \partial_j, \partial_k$ respectively we have

$$\nabla_{\partial k}\, h(\partial_i,\partial_j) = \overset{\perp}{\nabla}_{\partial k}\, h(\partial_i,\partial_j) - h(\nabla_{\partial_k}\partial_i,\partial_j) - h(\partial_i,\nabla_{\partial_k}\partial_j) = 0$$

$$\Longrightarrow (\nabla_{\partial_k} h)(\partial_i,\partial_j) = \overset{\perp}{\nabla}_{\partial k}\, h(\partial_i, \partial_j)$$

$$\left(\text{since } \nabla_{\partial i}\, \partial_j = 0\right).$$

which shows that $h(\partial_i,\partial_j)$ is global parallel normal field.

Q.E.D.

Lemma 3.2.4 : Let $f : M^n_j \longrightarrow R^m_k$ be an isometric immersion. If global normal field N is parallel with respect to the normal connection, then A_N commutes with A_ξ for all normal vector ξ .

Proof : We have Ricci equation

$$\overset{\perp}{R}(X,Y,N,\xi) = R(X,Y,N,\xi) - \frac{1}{2}\, g\,([A_N, A_\xi](X),Y)$$

Since N is parallel then

$$R\,(X,Y,N) = \overset{\perp}{\nabla}_X \overset{\perp}{\nabla}_Y N - \overset{\perp}{\nabla}_Y \overset{\perp}{\nabla}_X N - \overset{\perp}{\nabla}_{[X,Y]} N = 0$$

Which implies that

$$g([A_N, A_\xi](X),Y) = 0 \quad \forall \; X,Y \; .$$

Now the metric g is non-degenerate on M, we get

$$[\, A_N, A_\xi \,]\,(X) = 0$$

or

$$A_N\, A_\xi(X) - A_\xi\, A_N(X) = 0$$

or

$$A_N\, A_\xi = A_\xi\, A_N \; .$$

This shows that A_N commutes with A for all normal vector ξ .

Q.E.D.

We have the following classification theorem concerned with isometric immersion $R^n \longrightarrow R_1^{n+k}$ with parallel second fundamental form.

<u>Theorem</u> 3.2.2 : If $f : R^n \longrightarrow R_1^{n+k}$ is an isometric immersion with parallel second fundamental form, then it has trivial normal connection, i.e., $R^{\perp} \equiv 0$.

<u>Proof</u>: In order to show that f has trivial normal connection, it is sufficient to show that for any two normal vector and ^{1} A and A commutes .

By Lemma 3.2.3 and Lemma 3.2.4 if ν is in first normal space $N^1(x)$, then by definition A_ν commutes with every other shape operator.

On the other hand if $\rho \in N^o(x)$, then $A_\rho \equiv 0$ by definition and A_ρ commutes with every other shape operator.

If metric restricted to the first normal space is non-degenerate, this is sufficient because the normal space equals to $N^o(x) \oplus N_1(x)$.

If metric restricted to $N^1(x)$ is degenerate, then $N^o(x) + N^1(x)$ is not the entire normal space. In this case there exist a unique light like vector δ in $N^1(x) \cap N^o(x)$. Choosing a light like vector $\hat{\delta}$ with $\langle \delta, \hat{\delta} \rangle = 1$ we see that $N^o(x)$, $N^1(x)$ and $\hat{\delta}$ span the normal space. Since $A_{\hat{\delta}}$ commutes with itself, it is clear that all shape operator commute.

Q.E.D.

<u>Lemma</u> 3.2.5 : If N is global parallel normal vector field associated to an isometric immersion of R^n_1 with parallel second fundamental form then A_N has constant entries with respect to a flat co-ordinate system on R^n.

<u>Proof</u> : Let $(x^1, \ldots, x^n)$ be global co-ordinate system such that $\nabla_{\partial_i} \partial_j = o$, $1 \le i, j \le n$. Since the immersion has $(\nabla_Z h)(X,Y) = 0$ therefore

$$0 = A_N (\nabla_{\partial_i} \partial_j) = \nabla_{\partial_i} (A_N \partial_j)$$

$$= \nabla_{\partial_i} (a_{1j} \partial_1 + a_{2j} \partial_2 + \ldots + a_{nj} \partial_n)$$

$$= (\partial_i a_{1j}) \partial_1 + \ldots + (\partial_i a_{nj}) \partial_n \quad .$$

Therefore $\partial_i a_{kj} = o \quad \forall \quad i, j, k.$

Q.E.D.

To classify isometric immersions with parallel second fundamental form from R_1^n to R_1^{n+2} or R_2^{n+2}, it is necessary to determine those from R^2, to R_1^4 or R_2^4.

For this we give the following exmaple.

<u>Example</u> : B-Scroll over the null cubic in R_1^3.

In R_1^3 take a null curve $x(s)$ with null frame $A(s)$, $B(s)$, $C(s)$ such that

$$\dot{x}(s) = A(s)$$

$$(A(s), A(s)) = (B(s), B(s)) = (A(s), C(s)) = (B(s), C(s)) = 0$$

and $\quad (C(s), C(s)) = 1$

$$(A(s), B(s)) = 1$$

If these satisfy the following system of equations

$$\begin{aligned}
\dot{A}(s) &= k_1(s)\, A(s) + k_2(s)\, C(s) \\
\dot{B}(s) &= k_1(s)\, B(s) \\
\dot{C}(s) &= -\, k_2(s)\, B(s)
\end{aligned}$$

then $f^1(U,s) = x(s) + U\, B(s)$ is Lorentz surface in R_1^3 called a B scroll over $x(s)$

If $k_1 \equiv 0$ and $k_2 \equiv 1$ then

$\dddot{A} \equiv 0$ and the curve is called null cubic C (cf. [5]).

In this case we have B scroll over null cubic C

<u>Lemma</u> 3.2.6 : If $f : R^2_j \longrightarrow R^4_j$ or R^4_{j+1}, $j = 0,1$ is an isometric immersion with parallel second fundamental form and the mean curvature vector $\eta = 0$, then the first normal space has dimension less than two.

<u>Proof</u>: Assume, that the first normal space is of two dimension, and let ξ_1, ξ_2 be an orthonormal basis of $N^1(x)$. Using Theorem 3.1.3, put A_{ξ_1} into the appropriate canonical form. Thus

$$A_{\xi_1} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}.$$

Since A_{ξ_2} commutes with A_{ξ_1}, it is of the form

$$\begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} \text{ or } \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}.$$

A calculation of h on the basis of $Tx(m)$ shows that the first normal spaces are at most one dimensional.

Q.E.D.

Now we state the following theorem which is the main result of this section.

<u>Theorem</u> 3.2.3 : If $f : R_1^2 \longrightarrow R_1^4$ is an isometric immersion with parallel second fundamental form, then upto a rigid motion, f has one of the following form:

i) f is totally geodesic ,

ii) f is product of one dimensional maps ,

iii) $f : R_1^2 \longrightarrow R_1^3 \subset R_1^4$ is a B-scroll over the null cubic C.

CHAPTER-IV

NULLITY DISTRIBUTION OF INDEFINITE IMMERSION

Kinetsu and Majid [1] have defined a distribution called the nullity distribution and using its properties they have developed the geometry of submanifolds of indefinite space forms. In this chapter we give a detail account of their study. It has also been noted that these distributions give rise to a Ricatti type of differential equations along the geodesic in a leaf, very naturally. Unfortunately we cannot report much on the behaviour of the solutions of these differential equations because this is yet an open question.

4.1. Nullity Distributions:

In this section we define and study the basic properties of Nullity distributions.

Definition 4.1.1 : If $f : M_s^n \longrightarrow \bar{M}$ is an isometric immersion, we define relative nullity space at $x \in M$, $T^o(x)$

$$T^o(x) = \{X \in Tx(M) : A_\xi X \equiv 0 \quad \forall \quad \xi \in N(x)\}$$

The orthogonal complement $(T^o(x))^\perp$ in $Tx(M)$ is denoted by $T^1(x)$.

The following results give the alternative discription of the nullity distribution.

Proposition 4.1.1 : $T^o(x) = \{X \in Tx(M) : h(X,Y) = o \quad \forall Y \in Tx(M)\}$

Proof: It is obvious since $(h(X,Y),\xi) = (A_{\xi}X,Y)$
$\forall\ X,Y \in Tx(m)$

Proposition 4.1.2: $T^1(x) = \text{Span}\{A_{\xi}Y\}$ for $\xi \in N(x)$ and $Y \in Tx(m)$.

Proof: Given any $\xi \in N(x)$, $Y \in Tx(m)$ and $X \in T^o(x)$

$$\langle X, A_{\xi}Y \rangle = o \quad \text{so} \quad A_{\xi}Y \in T^1(x) .$$

Now suppose $Z \in Tx(m)$ satisfied

$$\langle Z, A_{\xi}Y \rangle = o \quad \forall\ \xi \in N(x) \text{ and } Y \in Tx(m) .$$

Then $Z \in T^o(x)$

$\Longrightarrow$ $h(Y,Z) = o$ By Proposition (4.1.1)
$\forall\ Y,Z \in T^o(x)$

$\Longrightarrow$ $[\text{Span}\ \{A_{\xi}Y\}]^{\perp} \subset T^o(x)$ so that
$[\text{Span}\ \{A_{\xi}Y\}] \supset T^1(x)$.

Q.E.D.

Definition 4.1.2 : The dimension $\nu(x)$ of $T^o(x)$ is called the relative nullity of the immersion at x. The minimum value of $\nu(x)$ on M is called the index of relative nullity and is denoted by ν_o.

If the index of relative nullity is constant, then we have the following basic result:

<u>Theorem</u> 4.1.1 : Assume that $\bar{M}$ is space form and let G denote the set of points in M where $v(x) = v_o(x)$. Then

1) G is an open subset of M,
2) $x \longrightarrow T^o(x)$, $x \in G$ is a differentiable and invalutive distribution in G,
3) The foliation T^o is totally geodesic in M and
4) each leaf of T^o is immersed as totally geodesic submanifold of $\bar{M}$.

<u>Definition</u> 4.1.3 : The distribution is said to be complementary in a neighbourhood of $y(t)$, $t \in [a,b)$, (where $y(t)$ is geodesic in a leaf of T^o) if $T^c(x) \oplus T^o(x) = Tx(m)$.

If at a fixed point $y(o)$, $T^o(y(o))$ is nondegenerate then $T^o(x)$ is nondegenerate for all points x near $y(o)$. Along geodesic $y(t)$ in a leaf of T^o, $T^o(y(t))$ remains nondegenerate since $T^o(y(t))$ is parallel along geodesic. In this case

$$T^c(x) = T^{\perp}(x).$$

If $T^o(y(o))$ is degenerate we use the following procedure. At $y(o)$, we choose pseudo orthonormal basis

$$\{L_1(o),\dots,L_r(o),E_1(o),\dots,E_{v_o-r}(o)\} \text{ of } T^o(y_o)$$

and

$$\{L_1(o),\dots,L_r(o),F_1(o),\dots,F_{n-v_o-r}(o)\} \text{ of } T^1(y(o))$$

so that

$$\langle L_i(o), L_j(o) \rangle = o = \langle L_i(o), E_k(o) \rangle = \langle L_i(o), F\ (o) \rangle$$

and

$E_k(o)$ and $F\ (o)$ form an orthonormal set. Adding the set $\{\hat{L}_1(o),\dots,\hat{L}_r(o)\}$ so that each $\hat{L}_i(o)$ is perpendicular to $E_k(o)$ and $F_\ell(o)$, $\langle \hat{L}_i(o), \hat{L}_j(o) \rangle = o$ and $\langle L_i(o), \hat{L}_j(o) \rangle = -\delta_{ij}$.

<u>Definition</u> 4.1.4 : In the neighborhood, of $y(t)$, we let Q be the projection defined by the decomposition $Tx(m) = T^o(x) + T^c(x)$

$$Q : Tx(m) \longrightarrow T^c(x) \qquad (4.1.4)$$

<u>Definition</u> 4.1.5 : For any $Y \in T^o$ and $X \in T\ M$ we define

$$C_Y X = -Q\ (\nabla_X Y) \qquad (4.1.5)$$

C, is called co-nullity operator.

Now we give the following technical lemma.

<u>Lemma</u> 4.1.1 : Let C and Q be defined by (4.1.4) and (4.1.5). If Y is in T^{o} and U and V are in $T\,M$ then

i) $Q\,(\nabla_Y U) = Q\,(\nabla_Y(QU))$,

ii) $Q\,(\nabla_{U-QU} Y) = 0$,

iii) $h\,(U,V) = h\,(QU,V)$,

iv) C is a tensor .

<u>Proof</u> : The proof of (i), (ii) and (iii) is trivial. Therefore we shall only give the proof of part (iv).

It is sufficient to show that

$$C_{\phi Y} = \phi\, C_Y U \quad \text{for} \quad \phi : M \longrightarrow R \ .$$

Now

$$\begin{aligned} C_{\phi Y} U &= -\,Q\,(\nabla_U(\phi Y)), && \big(\text{by definition}\big) \\ &= -\,Q\,((U\phi)\,Y + \phi\,(\nabla_U Y)), && \big(\nabla \text{ is connection in } TM\big) \\ &= -\,Q\,(\phi(\nabla_U Y)), && \big(\text{since } U\phi \in R \\ & && \quad (U\phi)\,Y \in T^{o}\big) \\ &= \phi\,(-Q(\nabla_U Y)) \\ &= \phi\,(C_Y U), && \big(\text{by definition}\big) \end{aligned}$$

This shows that C is tensor

Q.E.D.

Next we define connection in T^C and obtain its properties .

<u>Definition</u> 4.1.6 : Connection ∇' in T^C is defined by

$$\nabla'_U V = Q (\nabla_U V), \qquad (4.1.6)$$

$U \in T M$,
$V \in T^C$.

Using connection ∇' we differentiate C,

$$(\nabla'_Y C_Y) X = \nabla'_Y (C_Y X) - C_Y (\nabla'_Y X) .$$

Now

$$\nabla'_Y C_Y X = Q (\nabla_Y C_Y X) , \quad \left(\text{by definition of } \nabla'\right)$$

$$= - Q (\nabla_Y \nabla_X Y) , \quad \left(\text{by def. of } C.\right) .$$

Similarly

$$- C_Y (\nabla'_Y X) = Q (\nabla_{\nabla_Y X} Y) .$$

Combining both terms, gives

$$\nabla'_Y (C_Y X) - C_Y (\nabla'_Y X) = -Q(R(Y,X) Y + \nabla_X \nabla_Y Y - \nabla_{\nabla_X Y} Y.$$

Next we claim that

$$Q\,(\nabla_{\nabla_X Y} Y) = C_Y\,(C_Y X)$$

In fact

$$C_Y(C_Y X) = C_Y(-Q\nabla_X Y) = -\,Q(\nabla_{C_Y X}\,Y),\ \Big(\text{by definition of } C\Big)$$

$$= Q\,(\nabla_{Q(\nabla_X Y)}\,Y),\ \Big(\text{by definition of } C\Big).$$

$$C_Y(C_Y X) = Q\,(\nabla_{\nabla_X Y}\,Y) \qquad \Big(\text{by Lemma } (4.1.1)\Big)$$

If Y is an extension of tangent vectors $\bar{\gamma}_t$ along a geodesic in T^o, then

$$(\nabla'_Y C_Y)X = Q(R(X,Y)Y) + C_Y^2\,X \tag{4.1.6'}$$

The following result relates the completeness of M^n with the geodesic completeness of the leaves of nullity distribution.

<u>Theorem</u> 4.1.2 : If $f : M^n_s \longrightarrow M^{n+k}_t\,(C)$ is an isometric immersion and M^n is complete then the relative nullity foliation is geodesically complete.

<u>Lemma</u> 4.1.2 : For any Z in $T_{\gamma_a}(M)$ there exist $Z_t \in T_{\gamma_t}(M)$ $a \leq t < b$ such that $Z_a = Z$ and

$$\nabla'_t\,(QZ_t) + C_{\vec{\gamma}_t}\,(QZ_t) = 0 \quad \text{for} \quad a \leq t < b.$$

Moreover, Z_t can be extended differentiably to $t = b$. Here ∇'_t stands for ∇'_{Y_t}.

4.2 . Leaves of Nullity Distribution:

In this section we obtain the condition under which geodesically complete hypersurface of connected Lorentzian space form is totally geodesic. It has been observed that the index of nullity plays an important role in this result.

Equation (4.1.6') is an equation of Ricatti type and can be expressed as

$$\frac{dC(t)}{dt} = C^2(t) + K(t) \tag{4.2.1}$$

where

$$C(t) = \begin{bmatrix} C^L_{ij}(t) & D^L_{iq}(t) \\ C^F_{pj}(t) & D^F_{pq}(t) \end{bmatrix}$$

and $K(t) = C < Y,Y > I_n$.

We start with

<u>Lemma</u> 4.2.1 : The set of vectors $\{X, C_1(X), \ldots, C_{v_o-1}(X)\}$ forms a v_o frame in $T^C(x)$ for $X (\neq o) \in T^C(x)$.

<u>Proof</u>: Let $\alpha X + \alpha_1 C_1(X) + \dots + \alpha_{v_o-1} C_{v_o-1}(X) = 0$

Then $\alpha_1 C_1(X) + \dots + \alpha_{v_o-1} C_{v_o-1}(X) = C_{\alpha_1 Y_1 + \dots + \alpha_{v_o-1} Y_{v_o-1}}(X)$

where we denote $C_i = C_{Y_i}$ $(i=1,\dots,v_o-1)$ for simplicity.

Let $Y_1,\dots,Y_{v_o-1}$ be a set of (v_o-1) linearly independent space like vectors in $T^o(x)$ such that span $\{Y_1,\dots,Y_{v_o-1}\}$ is positive definite.

$$C_{\alpha_1 Y_1 + \dots + \alpha_{v_o-1} Y_{v_o-1}}(X) = -\alpha\,(X) .$$

Hence $-\alpha$ is real eigenvalue of $C_{\alpha_1 Y_1 + \dots \alpha_{v_o-1} Y_{v_o-1}}$

$$\alpha_1 Y_1 + \dots + \alpha_{v_o-1} Y_{v_o-1} = 0$$

$\Longrightarrow$ $\alpha_1 = \dots = \alpha_{v_o-1} = 0$ and $\alpha = 0$.

Q.E.D.

The following are the main results.

<u>Theorem</u> 4.2.1 : Let M_1^n be a geodesically complete connected submanifold of $\bar{M}_1^{n+k}(C)$, $C > 0$. If the index of relative nullity $v_o > V_n$ then M_1^n is totally geodesic in $\bar{M}_1^{n+k}(C)$ and $v_o = n$.

<u>Lemma</u> 4.2.2 : Let $(,)$ be a symmetric bilinear form on $n-v_0$ dimensional vector space V ~~ovdr~~ R with signature (m_1, m_2, m_3) if $m_1 \neq m_2$ and $T : V \longrightarrow V$ is symmetric linear operator with respect to $(,)$. Then T has a real eigen value.

<u>Note:</u> A symmetric bilinear form of signature (m_1, m_2, m_3) has $m_1(-1)$'s, $m_2(+1)$'s and $m_3 0$'s in the canonical form

<u>Lemma</u> 4.2.3 : Let $f : M_1^n \longrightarrow \bar{M}_1^{n+p}(C)$ be an isometric immersion between two Lorentizan manifolds, where $\bar{M}_1^{n+p}(C)$ is Lorentzian space form of positive curvature C. If the Ricci curvature S of M_1^n satisfies $S(X,X) \geq (n-1) C \langle X,X \rangle$ for all space like vectors, then $h(X,Y)$ is positive semi definite or Lorentzian.

Now define

$$K(X,Y) = \sum_{j=1}^{p} \langle A_j^2 X,Y \rangle \quad \text{on} \quad T^C(X) .$$

<u>Lemma</u> 4.2.4 : If $T^C(x)$ is positive defenite than K restricted to $T^C(x)$ is positive definite.

<u>Proof:</u> $K(X,Y) = \sum_{j=1}^{p} \langle A_j^2 X,X \rangle$

$$= \sum_{j=1}^{p} \langle A_j X, A_j X \rangle$$

If $K(X,X) = 0$ than $A_j X = 0$ for $j = 1,\ldots,p$. Since $T^c(x) = T^1(x) = \mathrm{span}\{A_j Y\}$ is positive definite. $\Longrightarrow$ $X \in T^o(x) \cap T^1(x)$, therefore $X = 0$, Hence K is positive definite.

Q.E.D.

<u>Lemma</u> 4.2.5 : If $T^c(x)$ is positive definite then $(\, , \,)$ is positive definite on $T^c(x)$.

<u>Proof</u>: For non-zero $e \in T^c(x)$, iff $h(e,e) = -1$ since this implies $\langle e,e \rangle \leq o$. Thus the form $(\, , \,)$ is the sum of a positive semi definite form and a positive definite form.

Q.E.D.

<u>Lemma</u> 4.2.6 : If $T^c(x)$ is degenerate subspace with respect to $\langle \, , \, \rangle$, then h is positive semi definite on $T^c(x)$ for $\dim T^c(x) > 1$.

<u>Proof</u>: $T^c(x)$ is positive semi definite subspace with respect to $\langle \, , \, \rangle$, by hypothesis. If there exist $e \in T^c(x)$ with $h(e,e) = -1$, then $\langle e,e \rangle = o$. By Lemma 4.2.3, there exist $g \in T^c(x)$ such that $h(g,g) = 1$ and $h(e,g) = 0$.

By hypothesis we have $\langle g,g \rangle > o$ and $\langle e,g \rangle = o$. For all t R $\langle g + te, g+te \rangle = \langle g,g \rangle > o$, but

$$\varsigma(X,Y) = h(X,Y) + C(n-1) < X,Y >$$

for convenience, put $C(n-1) = \tau$

$$(g+te,\ g+te) = h(g+te,\ g+te) + \tau < g+te,\ g+te >$$

$$= h\ (g,g) + h(te\ te,\ g+te) + 2h(g,te)$$

$$+ \tau\,[\langle g,g\rangle + t^2 \langle e,e\rangle + 2 \langle g,te\rangle]$$

$$= h\ (g,g) - t^2 + \tau\langle g,g\rangle$$

$$= 1 - t^2 + \tau\langle g,g\rangle > \tau\langle g,g\rangle .$$

This is a contradiction.

Q.E.D.

<u>Lemma</u> 4.2.7 : If $T^C(x)$ is degenerate subspace with respect to $\langle\ ,\ \rangle$ and if $\dim T^C(x) > 1$, then K is positive semi definite on $T^C(x)$ and for some $Y \in T^C(x)$, $K(Y,Y) > o$.

<u>Lemma</u> 4.2.8 : If M_1^n is a hypersurface in $\bar{M}_1^{n+1}(C)$, T^1 is Lorentizian and $n-v_o > 2$, then the signature (m_1,m_2,m_3) of $(\ ,\ )$ has $m_2 > m_1$.

Collecting all these lemmas now we are in position to state the following fundamental result of this chapter.

<u>Theorem</u> 4.2.2 : Let $f : M_1^n \longrightarrow \bar{M}_1^{n+p}(C)$ be an isometric immersion between two Lorentzian manifolds where $\bar{M}_1^{n+p}(C)$ is

the Lorentzian space form of positive curvature C. Suppose that the Ricci curvature S on M_1^n satisfies
$S(X,X) \geq (n-1) C \langle X,X \rangle$ for all space like vectors X.

i) if $T^0(x)$ is Lorentzian for some $x \in G$, then the index of relative nullity is either 0 or η.

ii) If $T^0(x)$ is degenrate for some $x \in G$, then the index of relative nullity is 0, 1, $n-1$ or n.

iii) If $T^0(x)$ is Riemannian for all $x \in G$ and if $p = 1$, then index of relative nullity is 0, $n-2$, $n-1$ or n.

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